8.

Trigonometric functions

- The graph of $y = \sin x$
- The graph of $y = \cos x$
- The graph of $y = \tan x$
- More about $y = \tan x$
- Positive or negative?
- Solving trigonometric equations
- Using the solve facility on a calculator
- The Pythagorean identity
- Angle sum and angle difference
- Miscellaneous exercise eight

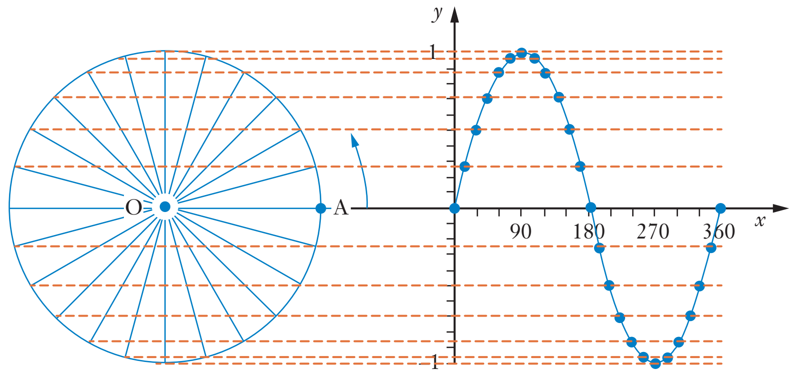
The graph of $y = \sin x$

In the diagram below, a circle of unit radius has been divided into sectors each of central angle 15°. The graph has been obtained by plotting on the y axis the height of point A above the centre of the circle as A moves around the unit circle in an anticlockwise sense, against the angle moved through on the x axis. The dots on the graph are plotted every 15°. From the unit circle definition of the sine of an angle, encountered in Chapter 1, it follows that the graph produced in this way is that of $y = \sin x$.

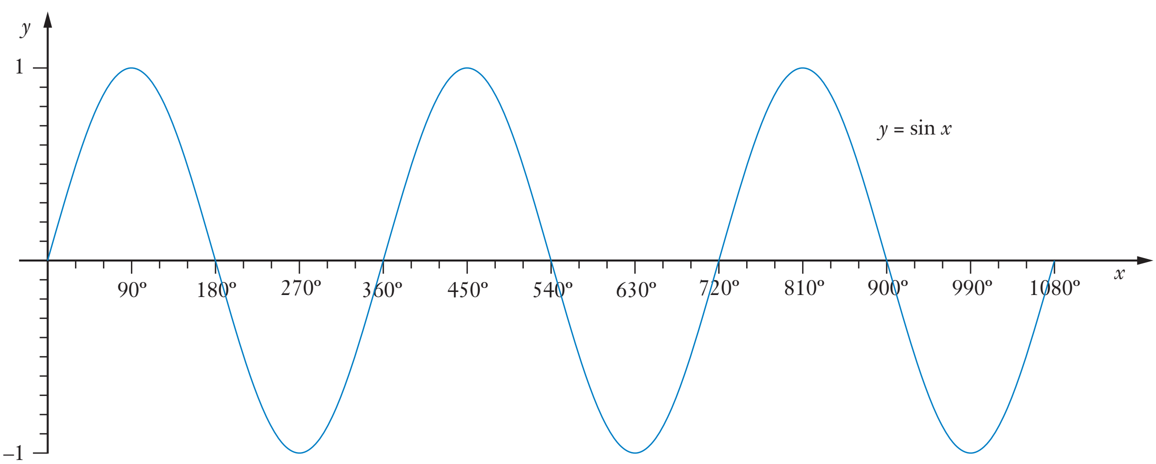
Sine and cosine curves

Sketching periodic functions-amplitude and period

Sketching periodic functions-phase and vertical shift



If, having completed one rotation of the circle, we were to continue moving point A around the circle the graph would repeat itself, as shown below for three rotations.



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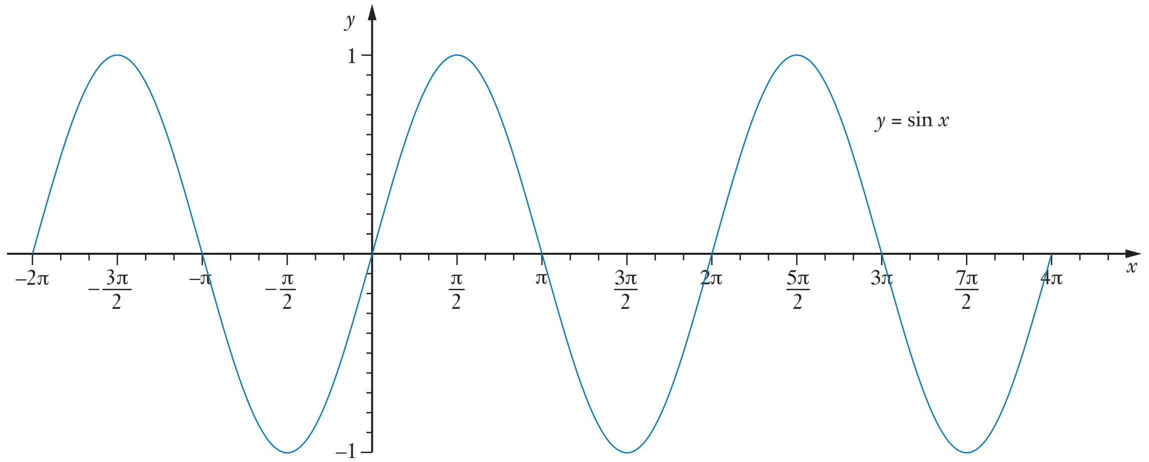
Alternatively angles could be shown in radians and negative angles could also be included, as shown below for $-2\pi \le x \le 4\pi$.



Whilst the graph shown above is for $-2\pi \le x \le 4\pi$, this restriction is made purely due to page width limitations. The reader should consider the graph of $y = \sin x$ continuing indefinitely to the left and the right.

Points to note:

- The graph of $y = \sin x$ repeats itself every 2π radians (or 360°).
  We say that the sine function is **periodic**, with **period** 2π.
  Thus $\sin (x \pm 2\pi) = \sin x$.
- We also say that the graph performs one **cycle** each period.
  Thus $y = \sin x$ performs one cycle in 2π radians (or 360°).
- Note that $-1 \le \sin x \le 1$.
- If we consider the above graph to have a 'mean' y-coordinate of $y = 0$ then the graph has a maximum value 1 above this mean value and a minimum value 1 below it. We say that $y = \sin x$ has an **amplitude** of 1.
- The graph passes the 'vertical line test', i.e. for each x value there is one and only one y value. Hence $y = \sin x$ is a function.
- Note that $\sin (-x) = -\sin x$. (Functions for which $f(-x) = -f(x)$ are called *odd* functions and are unchanged under a 180° rotation about the origin. As is the case for $y = x^n$ for *odd* values of n.)

Display the graph of $y = \sin x$ on your calculator and confirm that it is as shown above.

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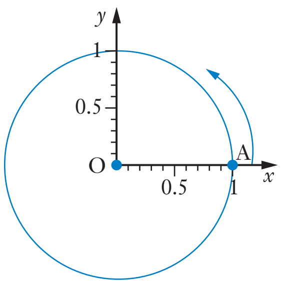
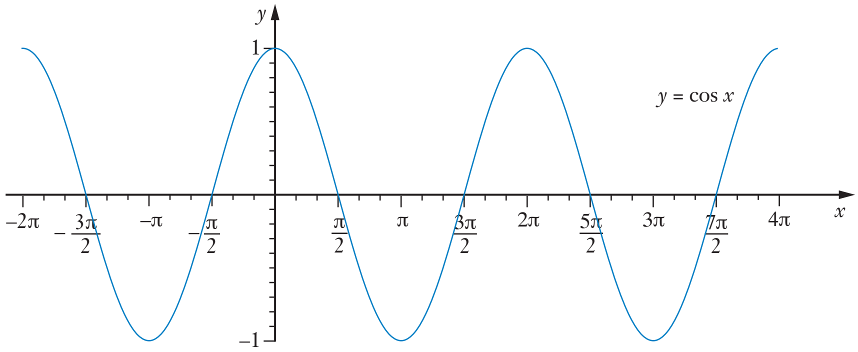
The graph of $y = \cos x$



Similar considerations of the x-coordinate of point A as it moves around the unit circle gives the graph of

$$y = \cos x,$$

shown below for $-2\pi \le x \le 4\pi$.



Points to note:

- The cosine function is **periodic**, with **period 2π**. Thus $\cos(x \pm 2\pi) = \cos x$.
- $y = \cos x$ performs one **cycle** in 2π radians (or 360°).
- Note that $-1 \le \cos x \le 1$.
- The graph of $y = \cos x$ has an **amplitude** of 1.
- Note that $\cos(-x) = \cos x$. (Functions for which $f(-x) = f(x)$ are called *even* functions and are unchanged under a reflection in the y-axis. As is the case for $y = x^n$ for *even* values of n.)
- If the above graph of $y = \cos x$ is moved $\frac{\pi}{2}$ units right, parallel to the x-axis, it would then be the same as the graph of $y = \sin x$. We say that $\sin x$ and $\cos x$ are $\frac{\pi}{2}$ out of **phase** with each other.

  It follows that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ and $\sin x = \cos\left(x - \frac{\pi}{2}\right)$.

Note

Whether an integer is even or odd is called the **parity** of the integer. Thus two odd numbers have the same parity. Similarly, whether a function is even, i.e. $f(-x) = f(x)$, or odd, i.e. $f(-x) = -f(x)$, is referred to as the parity of the function.

A function does not have to be even or odd. Many functions are neither even nor odd, (and indeed the function $f(x) = 0$ can be regarded as being *both* even and odd).

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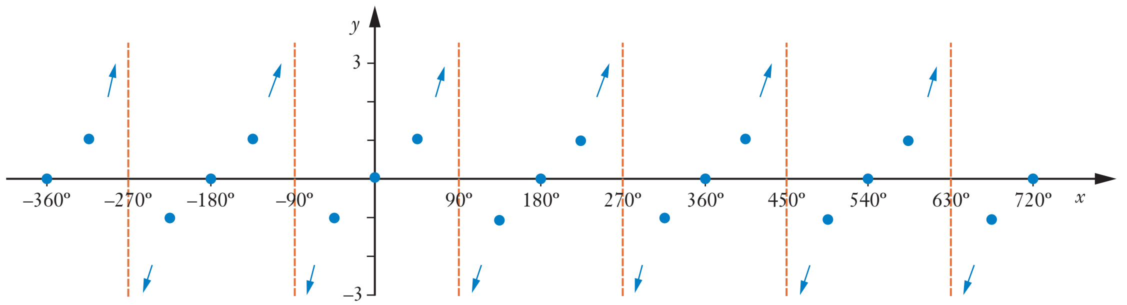
The graph of $y = \tan x$

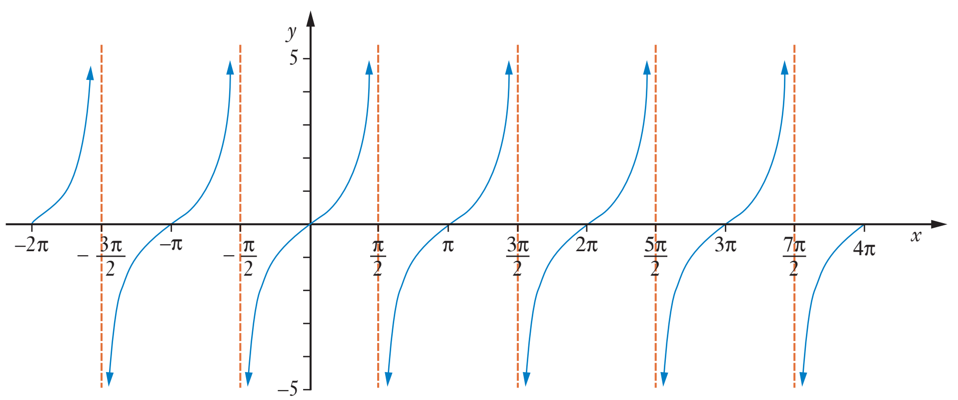
The fact that $\tan x = \dfrac{\sin x}{\cos x}$ was justified for right triangles in the *Preliminary work*.

If we assume this relationship to be true for all angles then it follows that $y = \tan x$ will

- equal zero for any value of x for which $\sin x = 0$, i.e. $x = 0°, \pm180°, \pm360°, \ldots$
- be undefined for any value of x for which $\cos x = 0$, i.e. $x = \pm90°, \pm270°, \ldots$
- equal 1 for any value of x for which $\sin x = \cos x$, eg. $x = 45°, 225°, 405°, \ldots$
- equal –1 for any value of x for which $\sin x = -\cos x$, eg. $x = 135°, 315°, \ldots$.



The completed graph of $y = \tan x$, for $-2\pi \le x \le 4\pi$, is shown below.



Note:

- Though the graph above is for $-2\pi \le x \le 4\pi$ the reader should consider the graph of $y = \tan x$ continuing indefinitely to the left and right.
- The graph repeats itself every π radians (or 180°).
  The **period** of the graph is π radians (or 180°). Thus $\tan(x \pm \pi) = \tan x$.
  The graph performs one **cycle** in π radians (or 180°).
- The term 'amplitude' is meaningless when applied to $y = \tan x$.
- The graph is such that $\tan(-x) = -\tan x$. (The tangent function is an *odd* function.)

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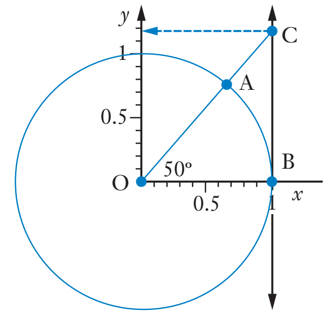
More about $y = \tan x$

The previous page developed the graph of $y = \tan x$ from the rule $\tan x = \dfrac{\sin x}{\cos x}$.



Alternatively we can use the unit circle to define the tangent of an angle directly. Consider some point A on the unit circle, centre at point O(0, 0). Point B is the point (1, 0). We will define the tangent of the angle AOB (shown as a 50° angle in the diagram) as the y-coordinate of the point where OA, continued as necessary, meets the vertical line through B. (This is point C in the diagram on the right and gives tan ∠AOB as just less than 1.2, i.e. $\tan 50° \approx 1.2$.)

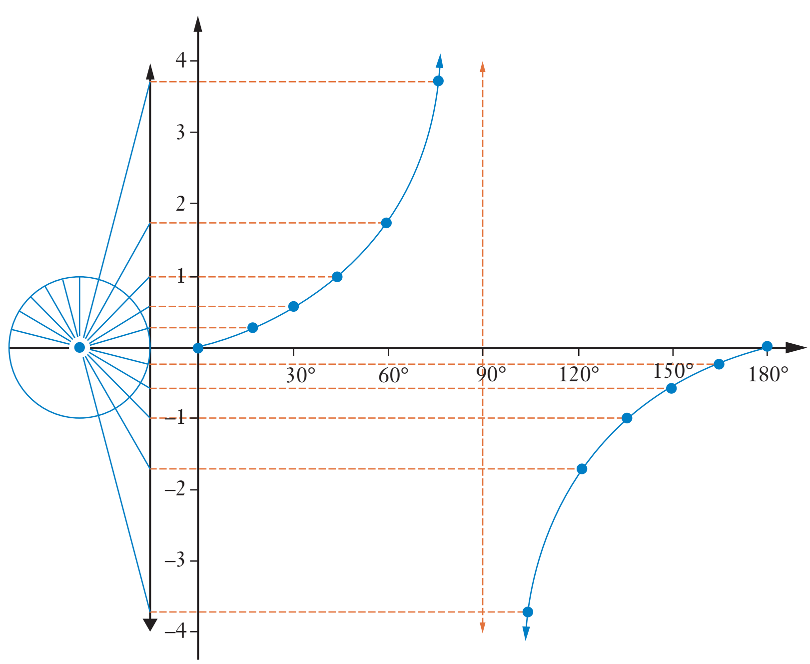
Note:

- The vertical through B is the tangent to the circle at point B so this definition of the trigonometric term *tangent* does involve a *tangent* to the unit circle.
- Applying the right triangle definition of tangent to triangle OBC, we also obtain tan ∠AOB = BC, so the two approaches are consistent.

The diagram below shows this definition used to produce the graph of $y = \tan x$ for x from 0° to 180°. The reader should confirm that this graph:

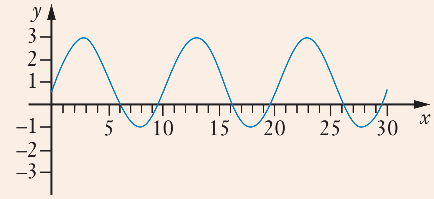
a is consistent with the graph produced previously using the unit circle definitions of $\sin x$ and $\cos x$ and the relationship $\tan x = \dfrac{\sin x}{\cos x}$

b would repeat every 180° thereafter.



EXAMPLE 1

State the amplitude and period of the function shown graphed on the right.



Solution

The graph appears to have a maximum value of 3, a minimum value of –1. These values are respectively 2 units above and 2 units below the 'mean' value of 1. The amplitude is 2.

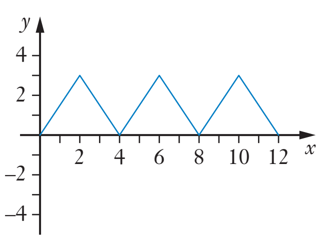
The graph appears to repeat itself every 10 units.

The period is 10.

Exercise 8A

State the period of each of the periodic functions shown graphed below.

1



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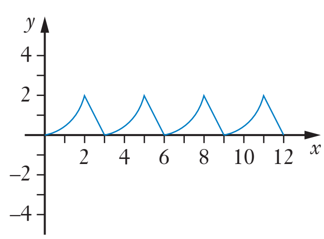

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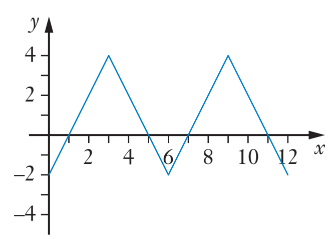

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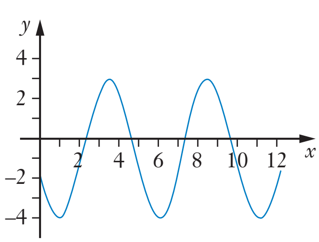

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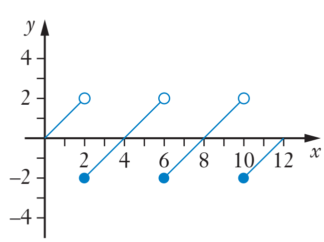

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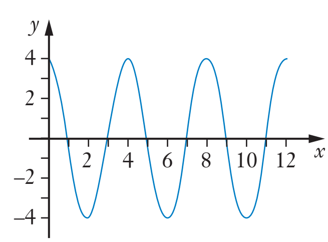

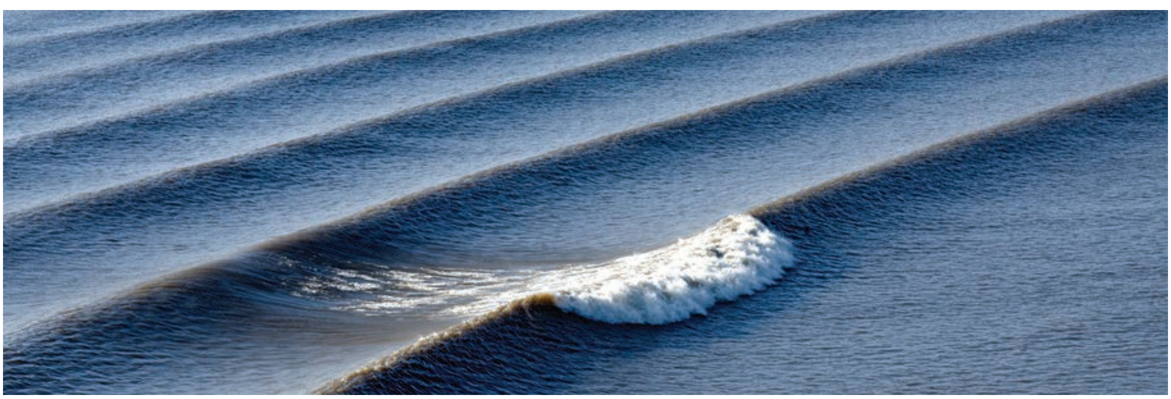

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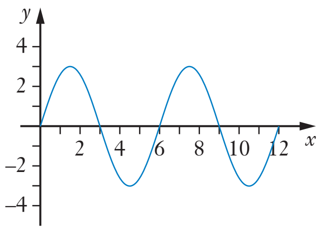
State the amplitude and period of each of the functions shown below.

7



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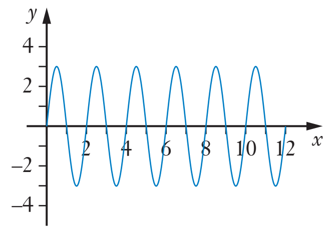

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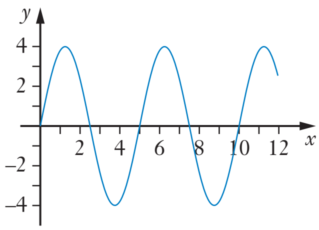

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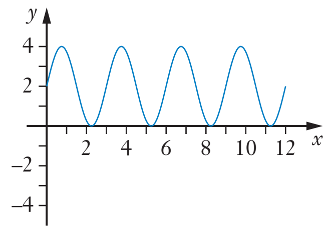

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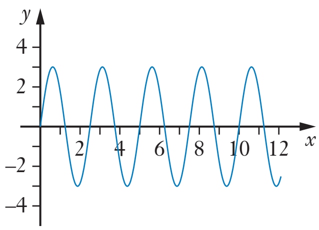

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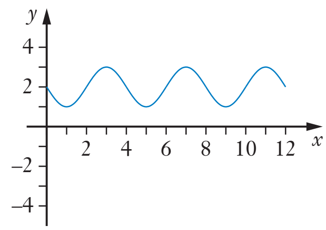

INVESTIGATION

- What effect does changing the value of a in $y = a \sin x$ have on the graph of the function? I.e. draw and compare $y = \sin x$, $y = 2 \sin x$, $y = 3 \sin x$, $y = -2 \sin x$ etc. and try to write a general statement regarding the effect altering a has on the graph of $y = a \sin x$.
- Similarly investigate changing the value of b in $y = a \sin (bx)$, for $b > 0$. I.e. for some fixed value of a, say 1, compare graphs of $y = 1 \sin x$, $y = 1 \sin 2x$, $y = 1 \sin 3x$, $y = 1 \sin 4x$ etc. and try to write a general statement regarding the effect altering b has on the graph of $y = a \sin bx$.
- Similarly investigate changing the value of c in $y = a \sin [b(x - c)]$ and d in $y = a \sin [b(x - c)] + d$.
- Similarly investigate the cosine and tangent functions.

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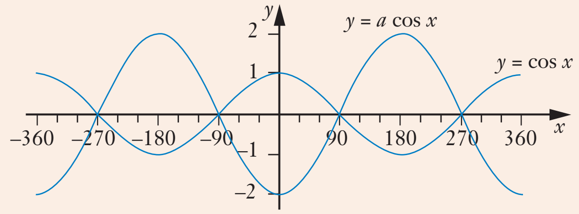
EXAMPLE 2

With x in degrees the graph on the right shows

$y = \cos x$

and $y = a \cos x$.



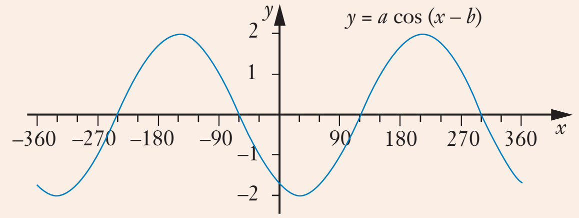
The second graph, below right, shows

$y = a \cos (x - b)$

for b in degrees and

$0 \le b \le 360$.

Determine a and b.



Solution

In the first diagram, the graph of $y = a \cos x$ is that of $y = \cos x$ stretched ↕ parallel to the y-axis until its amplitude is 2, and then reflected in the x-axis. This effect will be achieved if $a = -2$.

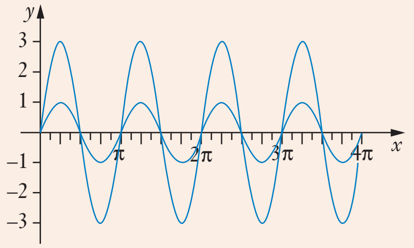
The second diagram shows the graph of $y = -2 \cos x$ moved 30° right. This is achieved if $b = 30$.

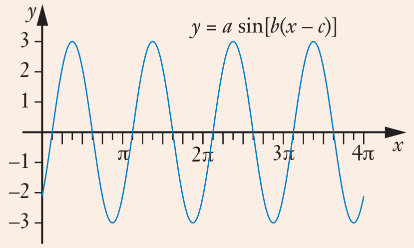
Hence $a = -2$ and $b = 30$.

EXAMPLE 3

With x in radians, the graph below left shows $y = \sin bx$ and $y = a \sin bx$.

The graph below right shows $y = a \sin [b(x - c)]$, $0 \le c \le \pi$. Find a, b and c.





Solution

The 'smaller' line in the first graph, with an amplitude of 1, must be $y = \sin bx$.

The line performs 2 cycles in the interval that $y = \sin x$ would perform 1. Hence $b = 2$.

(Alternatively we could say: $y = \sin bx$ has a period of $\frac{2\pi}{b}$. The given graph has a period of π.

Thus $\frac{2\pi}{b} = \pi$, which again gives $b = 2$.)

The other line in the first graph, is $y = \sin 2x$ stretched ↕ parallel to the y-axis until its amplitude is 3. Hence $a = 3$.

The second graph shows $y = 3 \sin 2x$ moved $\frac{\pi}{8}$ units right. Hence $a = 3$, $b = 2$ and $c = \frac{\pi}{8}$.

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EXAMPLE 4

Sketch the graph of $y = 2\tan\frac{x}{2}$ for $0 \le x \le 4\pi$ and then check the reasonableness of your sketch by viewing the graph of the function on your calculator.

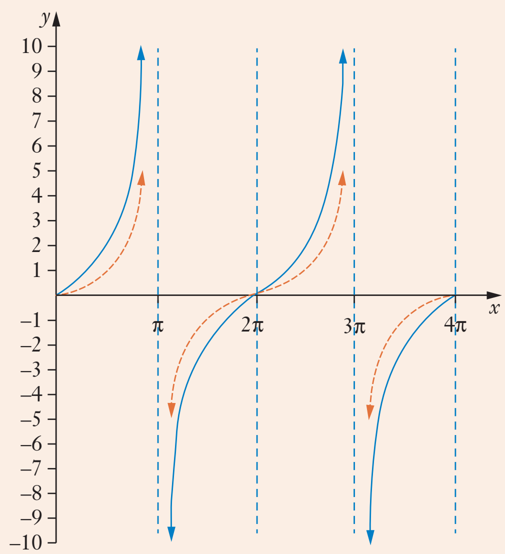
Solution

The graph of $y = \tan\frac{x}{2}$ will perform half of a cycle in the interval that $y = \tan x$ would perform 1 cycle, i.e. π radians. Thus $\tan\frac{x}{2}$ has a period of 2π radians.

Also (remembering that $\tan\frac{\pi}{4} = 1$) when $x = \frac{\pi}{2}$, $y = 1$.

The graph of $y = \tan\frac{x}{2}$ is as shown by the red broken lines in the diagram on the right.

Stretching $y = \tan\frac{x}{2}$ parallel to the y-axis ($\updownarrow$), until distances from the y-axis are doubled, will give the graph of $y = 2\tan\frac{x}{2}$, as shown by the blue solid lines in the graph. (The reader should check the reasonableness of the sketch by comparing it to that from a calculator display.)



Exercise 8B

Attempt the following without the assistance of a graphic calculator, then use your calculator to check your answers if you wish.

1 State the amplitude of each of the following.

a $y = \sin x$ **b** $y = 2\cos x$ **c** $y = 4\cos x$

d $y = -3\sin 2x$ **e** $y = 2\cos\left(x + \frac{\pi}{2}\right)$ **f** $y = -3\sin(x - \pi)$

g $y = 5\cos(x - 2)$ **h** $y = -3\cos(2x + \pi)$

2 State the period of each of the following for x in degrees.

a $y = \sin x$ **b** $y = \tan x$ **c** $y = 2\sin x$

d $y = \sin 2x$ **e** $y = \cos\frac{x}{2}$ **f** $y = \cos 3x$

g $y = 3\tan 2x$ **h** $y = 3\sin\left(\frac{x - 60°}{3}\right)$ **i** $y = 5\sin[2(x - 30°)]$

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3 State the period of each of the following for x in radians.

a $y = \cos x$ **b** $y = \tan x$ **c** $y = 3\cos x$

d $y = 2\cos 4x$ **e** $y = 2\tan 3x$ **f** $y = \frac{1}{2}\sin 3x$

g $y = 3\sin\left(\frac{x}{2}\right)$ **h** $y = 2\cos(2x - \pi)$ **i** $y = 2\sin(4\pi x)$

4 Determine the coordinates of any maximum and minimum points on each of the following functions for $0 \le x \le 2\pi$.

a $y = \sin x$ **b** $y = 2 + \sin x$ **c** $y = -\sin x$

d $y = \sin(2x) + 3$ **e** $y = \sin\left(x - \frac{\pi}{4}\right) + 3$

5 State the greatest value each of the following can take and the smallest positive value of x (in degrees) that gives this maximum value.

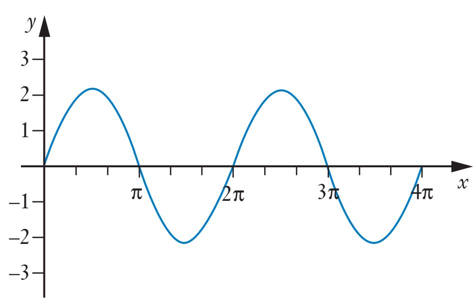
a $3\sin x$ **b** $2\sin(x - 30°)$ **c** $2\sin(x + 30°)$ **d** $-3\sin x$

6 State the greatest value each of the following can take and the smallest positive value of x (in radians) that gives this maximum value.

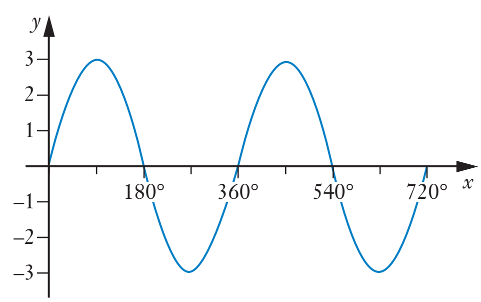
a $3\sin 2x$ **b** $-5\sin x$ **c** $2\cos\left(x + \frac{\pi}{6}\right)$ **d** $3\cos\left(x - \frac{\pi}{6}\right)$

7 Each of the following graphs has an equation of the form $y = a\sin x$. State the value of a in each case.

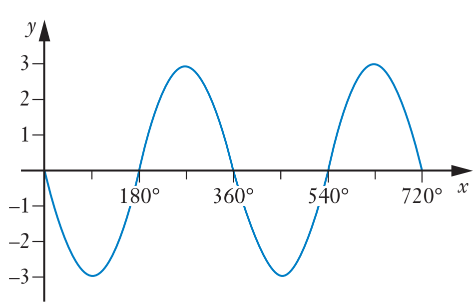
a



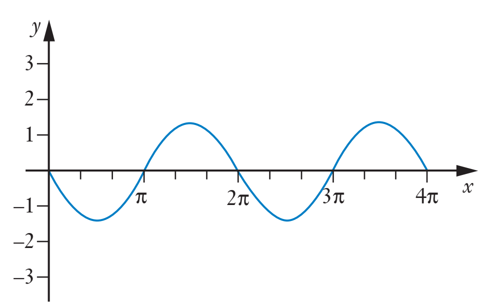
b



c



d



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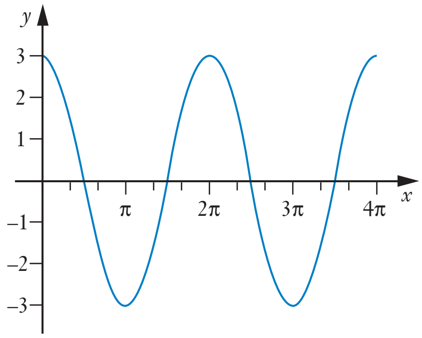
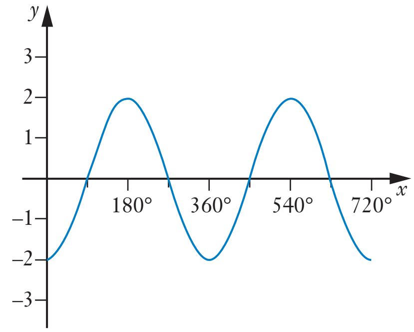
8 Each of the following graphs has an equation of the form $y = a \cos x$.

State the value of a in each case.

a



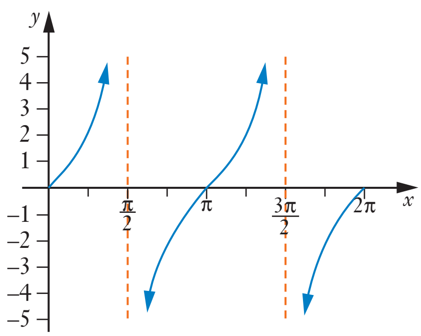
b



9 Each of the following graphs has an equation of the form $y = a \tan x$.

State the value of a in each case.

a



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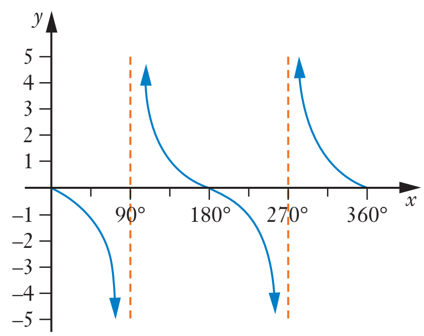
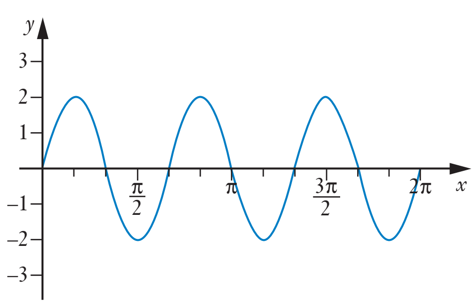


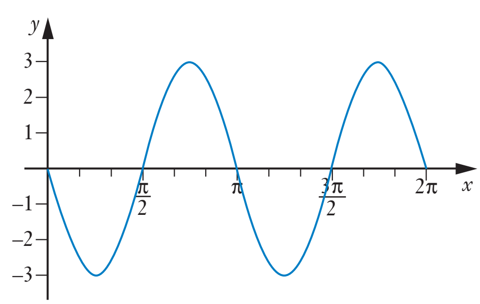
10 Each of the following graphs has an equation of the form $y = a \sin bx$.

State the values of a and b in each case.

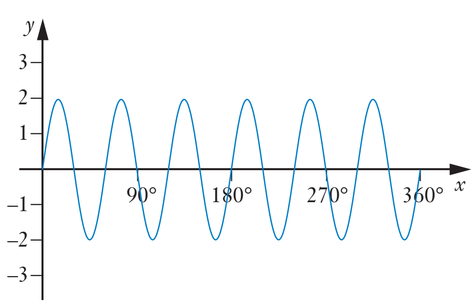
a



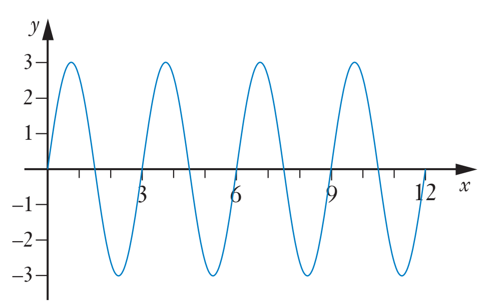
b



c



d



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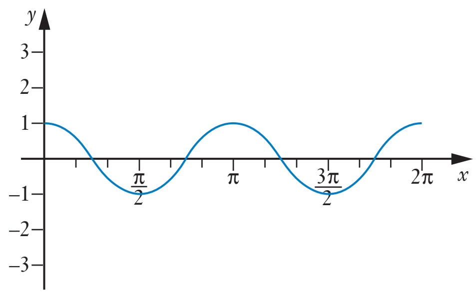
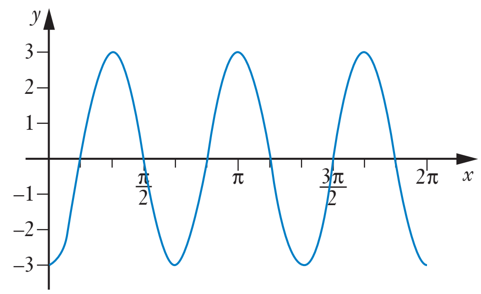
11 Each of the following graphs has an equation of the form $y = a \cos bx$.

State the value of a and b in each case.

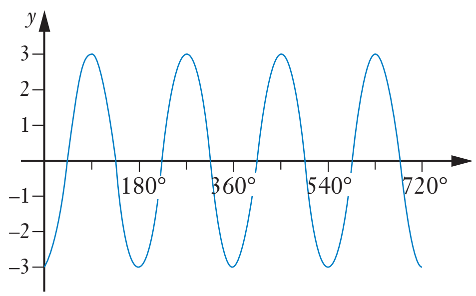
a



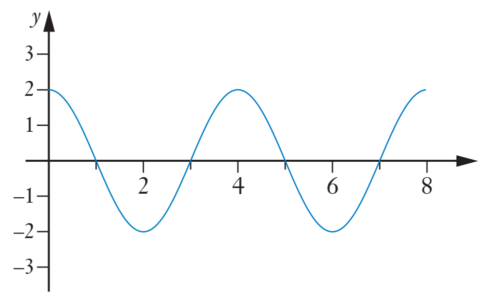
b



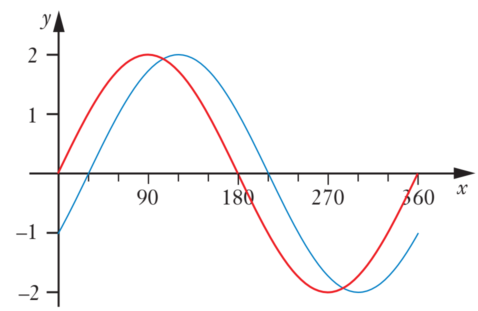
c



d



12 The graph on the right shows $y = a \sin x°$ and $y = a \sin (x - b)°$, with a the same integer in both equations, b a multiple of ten and one of the lines shown as a red line.



a Find the value of a and the two smallest possible positive values of b.

b Give an equation for the blue line in the form $y = c \sin (x - d)°$ with c a negative integer, d a multiple of ten and $0 \le d \le 360$.

13 a State the period and amplitude of $y = 3 \cos (\pi x)$.

b Sketch the graph of $y = 3 \cos (\pi x)$ for $0 \le x \le 6$.

14 a State the period and amplitude of $y = -5 \sin\left(\frac{\pi x}{2}\right)$.

b Sketch the graph of $y = -5 \sin\left(\frac{\pi x}{2}\right)$ for $0 \le x \le 8$.

15 Sketch both of the following on a single pair of axes, with $0 \le x \le 360°$.

a $y = 2 \tan x$

b $y = 2 \tan (x + 45°)$

16 Sketch both of the following on a single pair of axes, with $0 \le x \le 2\pi$.

a $y = 3 \sin 2x$

b $y = 3 \sin\left(2x - \frac{\pi}{3}\right)$, i.e. $y = 3 \sin\left[2\left(x - \frac{\pi}{6}\right)\right]$

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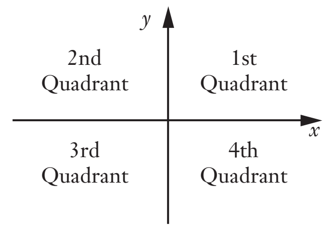
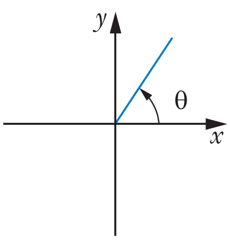
Positive or negative?



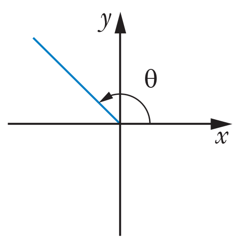
As we saw in Chapter 7, the x- and y-axes divide the coordinate plane into four regions called **quadrants**. These four quadrants are numbered as shown on the right.

- Any points lying in the 1st or 4th quadrants will have a positive x-coordinate.
- Any points lying in the 2nd or 3rd quadrants will have a negative x-coordinate.

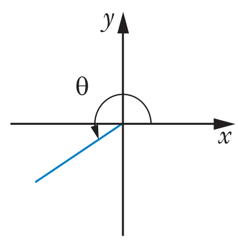
With our unit circle definition for the cosine of an angle involving the x-coordinate of a point moving around the circle, it follows that angles with their *initial ray* along the positive x-axis, and their *terminal ray* lying in either the 1st or 4th quadrants, will have positive cosines and any with their terminal ray lying in the 2nd or 3rd quadrants will have cosines that are negative.



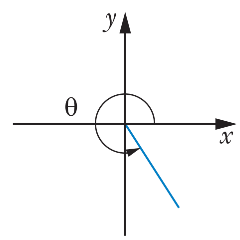
cos θ is positive



cos θ is negative

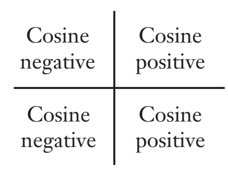


cos θ is negative

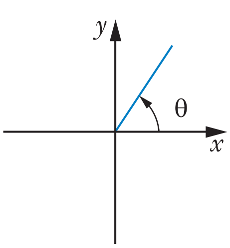


cos θ is positive

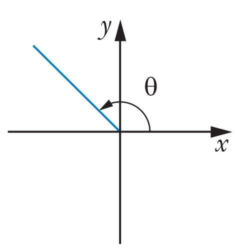
This positive or negative nature of the cosine function can be summarised as shown in the diagram on the right.



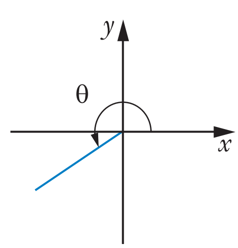
Similarly, remembering that the unit circle definition of the sine of an angle involves the y-coordinate of a point moving around the circle, the situation for sine is as follows:



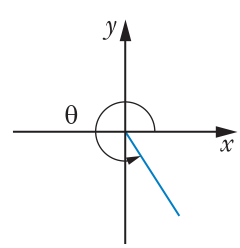
sin θ is positive



sin θ is positive

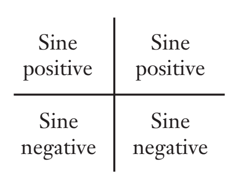


sin θ is negative



sin θ is negative

This positive or negative nature of the sine function can be summarised in the diagram on the right.



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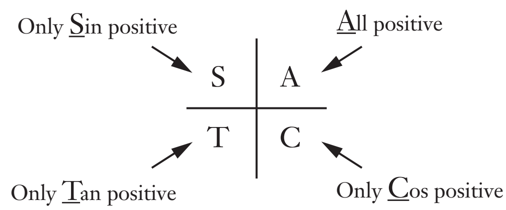
From $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ it follows from these facts about sine and cosine that the positive and negative nature of $\tan\theta$ will be as shown in the diagram on the right.

Tangent negative	Tangent positive
Tangent positive	Tangent negative

These facts regarding the positive and negative nature of sine, cosine and tangent are summarised below.



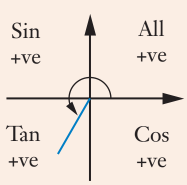
EXAMPLE 5

Without the assistance of a calculator state whether each of the following are positive or negative.

a $\sin 240°$ **b** $\cos 170°$ **c** $\tan(1.2\pi)$ **d** $\sin\left(-\dfrac{\pi}{3}\right)$

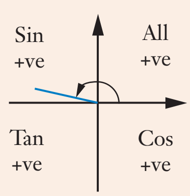
Solution

a



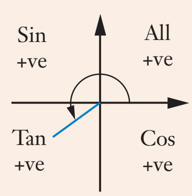
$\sin 240°$ is negative

b



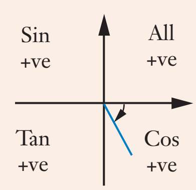
$\cos 170°$ is negative

c



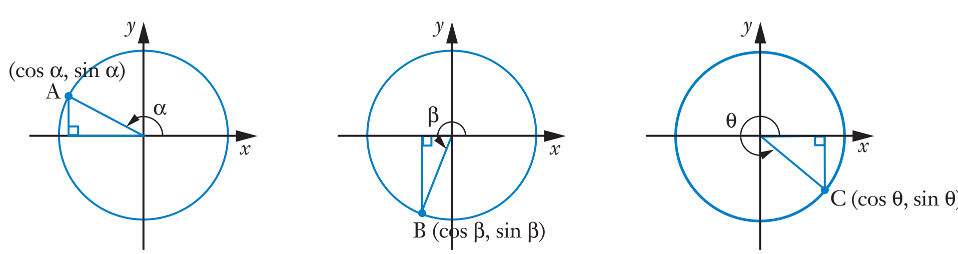
$\tan(1.2\pi)$ is positive

d



$\sin\left(-\dfrac{\pi}{3}\right)$ is negative

Consider angles α, β and θ as shown below.



In each case, the right-angled triangle made with the x-axis could be re-drawn in the first quadrant as shown on the next page.

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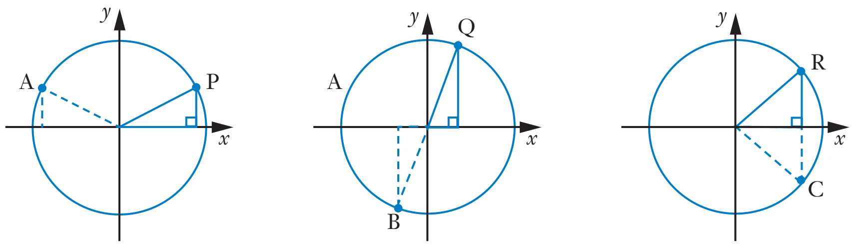


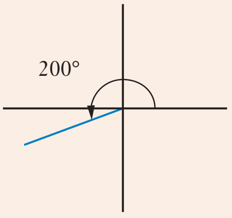
The x- and y-coordinates of P, Q and R may differ from those of A, B and C only in sign.

e.g. x-coordinate of P = –x-coordinate of A,
y-coordinate of P = y-coordinate of A.

Thus the sine (or cosine or tangent) of any angle will equal the sine (or cosine or tangent) of the acute angle made with the x-axis together with the appropriate sign.

EXAMPLE 6

Express sin 200° in terms of the sine of an acute angle.



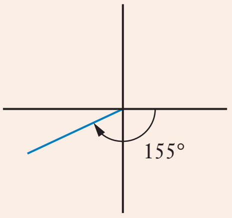
Solution

An angle of 200° makes 20° with the x-axis and lies in the 3rd quadrant, where the sine function is negative.

Thus sin 200° = –sin 20°.

EXAMPLE 7

Express tan (–155°) in terms of the tangent of an acute angle.



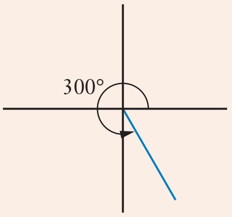
Solution

An angle of –155° makes 25° with the x-axis and lies in the 3rd quadrant, where the tangent function is positive.

Thus tan (–155°) = tan 25°.

EXAMPLE 8

Find the exact value of cos 300°.



Solution

An angle of 300° makes 60° with the x-axis and lies in the 4th quadrant, where the cosine function is positive. Thus cos 300° = cos 60°

$$= \frac{1}{2}.$$

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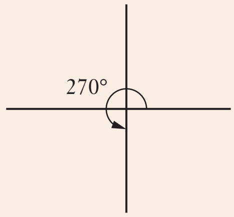
EXAMPLE 9

Find the exact value of sin 270°.



Solution

An angle of 270° makes 90° with the x-axis and lies on the boundary between the 3rd and 4th quadrant. In both of these quadrants the sine function is negative.

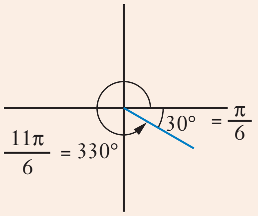
$$\text{Thus } \sin 270° = -\sin 90° = -1.$$

EXAMPLE 10

Give the exact value of $\tan \frac{11\pi}{6}$.

Solution

An angle of $\frac{11\pi}{6}$ makes $\frac{\pi}{6}$ with the x-axis and lies in the 4th quadrant, where the tangent function is negative.



$$\tan \frac{11\pi}{6} = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

(Or, expressed with a rational denominator, $-\frac{\sqrt{3}}{3}$.)

Exercise 8C

(Without the assistance of a calculator.)

For each of the following state whether positive or negative.

1 tan 190° **2** cos 310° **3** tan (–190°) **4** sin (–170°)

5 sin 555° **6** cos 190° **7** $\tan \frac{\pi}{10}$ **8** $\sin \frac{4\pi}{5}$

9 $\cos \frac{\pi}{10}$ **10** $\sin \left(-\frac{\pi}{5}\right)$ **11** $\cos \frac{9\pi}{10}$ **12** $\tan \frac{13\pi}{5}$

Express each of the following in terms of the sine of an acute angle.

13 sin 140° **14** sin 250° **15** sin 340° **16** sin 460°

17 $\sin \frac{5\pi}{6}$ **18** $\sin \frac{7\pi}{6}$ **19** $\sin \frac{11\pi}{5}$ **20** $\sin \left(-\frac{\pi}{5}\right)$

Express each of the following in terms of the cosine of an acute angle.

21 cos 100° **22** cos 200° **23** cos 300° **24** cos (–300°)

25 $\cos \frac{4\pi}{5}$ **26** $\cos \frac{9\pi}{10}$ **27** $\cos \frac{11\pi}{10}$ **28** $\cos \frac{21\pi}{10}$

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Express each of the following in terms of the tangent of an acute angle.

29 $\tan 100°$ **30** $\tan 200°$ **31** $\tan(-60°)$ **32** $\tan(-160°)$

33 $\tan \frac{6\pi}{5}$ **34** $\tan\left(-\frac{6\pi}{5}\right)$ **35** $\tan \frac{11\pi}{5}$ **36** $\tan\left(-\frac{21\pi}{5}\right)$

Give the exact value of each of the following.

37 $\sin 300°$ **38** $\tan 210°$ **39** $\cos 240°$ **40** $\cos 270°$

41 $\sin 180°$ **42** $\cos 390°$ **43** $\sin(-135°)$ **44** $\cos(-135°)$

45 $\sin \frac{7\pi}{6}$ **46** $\cos \frac{7\pi}{6}$ **47** $\tan \frac{7\pi}{6}$ **48** $\sin \frac{7\pi}{4}$

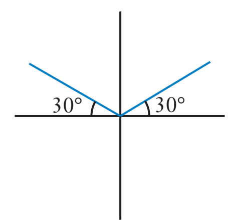
49 $\cos\left(-\frac{7\pi}{4}\right)$ **50** $\tan(6\pi)$ **51** $\sin \frac{5\pi}{2}$ **52** $\cos\left(-\frac{7\pi}{3}\right)$

Solving trigonometric equations

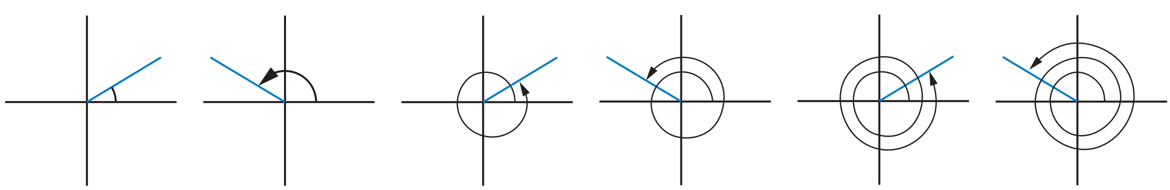
WS Trigonometric equations

Suppose we are asked to find an angle, x, such that $\sin x = 0.5$. With $\sin x$ being positive we know that any solutions must lie in the 1st and 2nd quadrants. The acute angle made with the x-axis must be 30° because, from our exact values, we know that $\sin 30° = 0.5$.

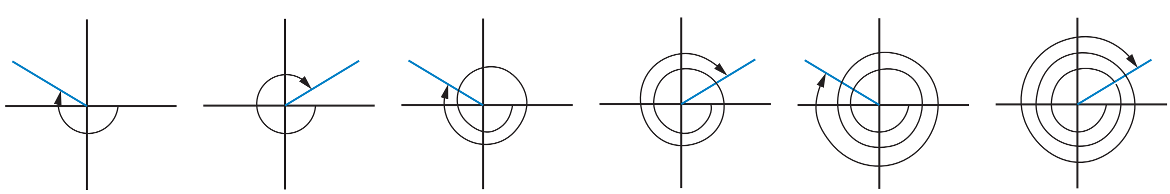
Thus, diagrammatically, the two possibilities for x are as shown on the right.



However, if there is no restriction on x, there are an infinite number of values of x that we can obtain from this diagram, and for all of these $\sin x = 0.5$. Twelve such values of x, six positive and six negative, are shown below. The reader should use a calculator to confirm that each of these values satisfy the requirement that $\sin x = 0.5$.



$x = 30°$ $x = 150°$ $x = 390°$ $x = 510°$ $x = 750°$ $x = 870°$



$x = -210°$ $x = -330°$ $x = -570°$ $x = -690°$ $x = -930°$ $x = -1050°$

Thus when asked to solve trigonometrical equations we will usually be given certain restrictions on the range of values the solutions can take.

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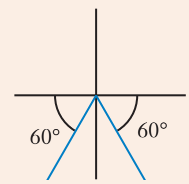
EXAMPLE 11

Without the assistance of a calculator, solve $\sin x = -\frac{\sqrt{3}}{2}$ for $0 \le x \le 360°$.

Solution

With the sine being negative, solutions must lie in the 3rd and 4th quadrants.

From our exact values we know that $\sin 60° = \frac{\sqrt{3}}{2}$.



Thus the solutions make 60° with the x-axis as shown diagrammatically on the right.

Using this diagram to obtain solutions in the required interval we have $x = 240°, 300°$.

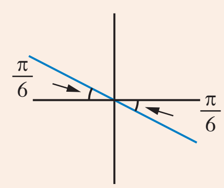
EXAMPLE 12

Without the assistance of a calculator solve $\tan x = -\frac{1}{\sqrt{3}}$ for $0 \le x \le 2\pi$.

Solution

With the tangent being negative, solutions must lie in the 2nd and 4th quadrants.

From our exact values we know that $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$.



Thus the solutions make $\frac{\pi}{6}$ radians with the x-axis as shown diagrammatically on the right.

Using this diagram to obtain solutions in the required interval gives: $x = \frac{5\pi}{6}, \frac{11\pi}{6}$.

Notice that in the previous example the solutions were given in radians because we were told in the question that $0 \le x \le 2\pi$ rather than $0 \le x \le 360°$.

Using the solve facility on a calculator

The equations of the above examples can be solved using the solve facility available on some calculators (with the calculator set to degrees or radians as appropriate for each question). Note that in the display on the right, the required interval in which we require solutions, is given.

$$\text{solve}\left(\sin(x) = \frac{-\sqrt{3}}{2}, x\right) | 0 \le x \le 360$$

$$x = 240 \text{ or } x = 300$$

$$\text{solve}\left(\tan(x) = \frac{-1}{\sqrt{3}}, x\right) | 0 \le x \le 2 \cdot \pi$$

$$x = \frac{5 \cdot \pi}{6} \text{ or } x = \frac{11 \cdot \pi}{6}$$

If no interval is stated the calculator will give a general solution involving some constant which, when suitable integer values are substituted for the constant, solutions can be obtained for any required interval. Questions requiring general solutions are not included in this text and we will only encounter equations for which a required interval is stated.

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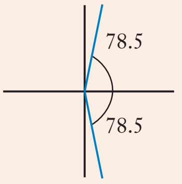
The following examples are solved without using the solve facility. Make sure that you can demonstrate your ability to solve trigonometric equations both with and without a calculator if required to do so.

EXAMPLE 13

Given that one solution to the equation $\cos x = 0.2$ is, correct to one decimal place, $x = 78.5°$, determine any other solutions the equation has for $-180° \le x \le 180°$, giving answers correct to one decimal place.

Solution

With the cosine of x being positive we know that solutions must lie in the 1st and 4th quadrants.



Hence for $-180° \le x \le 180°$ the other solution to the equation is $x = -78{\cdot}5°$ (correct to one decimal place).

EXAMPLE 14

Use the information that $\sin 36.9° = 0.6$ to determine all solutions to $5 \sin x = 3$ in the interval $0 \le x \le 720°$.

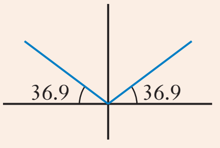
Solution

$$5 \sin x = 3,$$

$$\therefore \quad \sin x = \frac{3}{5}$$

$$= 0.6$$

Thus the solutions must lie in the 1st and 2nd quadrants and make 36.9° with the x-axis as shown diagrammatically on the right.



Using this diagram to obtain solutions in the required interval gives: $x = 36.9°, 143.1°, 396.9°, 503.1°$

EXAMPLE 15

Given that if $x = 1.11$ radians then $\tan x = 2$, solve the equation $\sin x = -2 \cos x$ for $-\pi \le x \le \pi$, giving answers in terms of π if necessary.

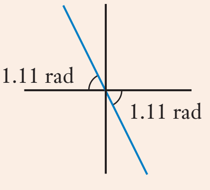
Solution

$$\sin x = -2 \cos x$$

Dividing both sides by $\cos x$ gives $\dfrac{\sin x}{\cos x} = -2\dfrac{\cos x}{\cos x}$,

i.e. $\tan x = -2$

Thus the solutions must lie in the 2nd and 4th quadrants and make 1.11 radians with the x-axis as shown diagrammatically on the right.



Using this diagram to obtain solutions in the required interval gives: $x = -1.11$ radians, $(\pi - 1.11)$ radians.

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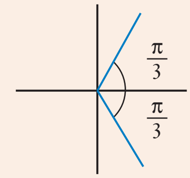
EXAMPLE 16

Solve $\cos 2x = 0.5$ for $0 \le x \le 2\pi$.

Solution

If $\cos 2x = 0.5$ then values of $2x$ must lie in the 1st and 4th quadrants and make $\frac{\pi}{3}$ radians with the x-axis.



Thus $\quad 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \qquad$ giving $\qquad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

Notice that in the previous example, to obtain all the solutions for x in the interval 0 to 2π we had to list values for $2x$ in the interval 0 to 4π. These values for $2x$, when divided by 2, gave values for x in the required interval.

Note

If we multiply sin x by itself we could write this as $(\sin x)(\sin x)$ or $(\sin x)^2$. However, to avoid having to write the brackets each time, we write this as $\sin^2 x$. This notation is evident in Example 17.

EXAMPLE 17

Solve the equation $2 \sin^2 x - 3 \sin x - 2 = 0$ for $0 \le x \le 4\pi$.

Solution

The equation is a quadratic in $\sin x$.

To solve the quadratic equation $\qquad 2y^2 - 3y - 2 = 0$

we look for two numbers which add to give -3 and multiply to give -4. I.e. $+1$ and -4.

We then rewrite the equation as $\qquad 2y^2 + 1y - 4y - 2 = 0$

Hence $\qquad y(2y + 1) - 2(2y + 1) = 0$

and so $\qquad (2y + 1)(y - 2) = 0$

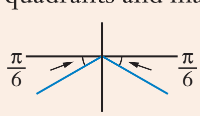
Thus, given the equation $\qquad 2 \sin^2 x - 3 \sin x - 2 = 0$

factorising gives $\qquad (2 \sin x + 1)(\sin x - 2) = 0$

Either $\qquad 2 \sin x + 1 = 0 \qquad$ or $\qquad \sin x - 2 = 0$,

i.e. $\qquad \sin x = -0.5 \qquad$ or $\qquad \sin x = 2$.

Solutions to $\sin x = -0.5$ must lie in the 3rd and 4th quadrants and make $\frac{\pi}{6}$ with the x-axis:



Thus $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$.

From our unit circle definition it follows that $-1 \le \sin x \le 1$.

Thus $\sin x = 2$ has no solution.

Thus for $0 \le x \le 4\pi$ the solutions are $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$.

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Exercise 8D

(Do this exercise without using the solve facility of a calculator.)

Solve the following for $0 \le x \le 360°$.

1 $\cos x = \frac{1}{2}$ **2** $\sin x = -\frac{1}{2}$ **3** $\tan x = 1$ **4** $\sin x = -\frac{1}{\sqrt{2}}$

Solve the following for $0 \le x \le 2\pi$.

5 $\sin x = \frac{1}{\sqrt{2}}$ **6** $\cos x = -\frac{1}{\sqrt{2}}$ **7** $\tan x = -1$ **8** $\tan x = \sqrt{3}$

Solve the following for $-180° \le x \le 180°$.

9 $\cos x = \frac{\sqrt{3}}{2}$ **10** $\sin x = -1$ **11** $\tan x = -\frac{1}{\sqrt{3}}$ **12** $\sin x = 0$

Solve the following for $-\pi \le x \le \pi$.

13 $\sin x = \frac{\sqrt{3}}{2}$ **14** $\cos x = -\frac{1}{2}$ **15** $\sin x = \frac{1}{2}$ **16** $\cos x = 0$

17 For x in radians, one solution of the equation $\tan x = 1.5$ is $x = 0.98$, correct to two decimal places. Hence determine, in terms of π, any other values of x in the interval $0 \le x \le 2\pi$ for which $\tan x = 1.5$.

18 Use the information that $\cos 63.9° = 0.44$ to determine solutions to

$$11 + 25 \cos x = 0$$

in the interval $-180° \le x \le 180°$.

Solve the following for x in the given interval.

19 $\tan 2x = \frac{1}{\sqrt{3}}$ for $0 \le x \le 180°$

20 $\cos 4x = \frac{\sqrt{3}}{2}$ for $0 \le x \le \pi$

21 $\sin 3x = \frac{1}{2}$ for $-90° \le x \le 90°$

22 $2\sqrt{3} \sin 2x = 3$ for $0 \le x \le 2\pi$

23 $2 \cos 3x + \sqrt{3} = 0$ for $0 \le x \le 2\pi$

24 $(\sin x + 1)(2 \sin x - 1) = 0$ for $0 \le x \le 2\pi$

25 $\sin^2 x = \frac{1}{2}$ for $0 \le x \le 360°$

26 $4 \cos^2 x - 3 = 0$ for $-\pi \le x \le \pi$

27 $(\sin x)(2 \cos x - 1) = 0$ for $-180° \le x \le 180°$

28 Solve $2 \cos^2 x + \cos x - 1 = 0$ for $-\pi \le x \le \pi$

29 Solve $\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$ for $0 \le x \le 2\pi$.

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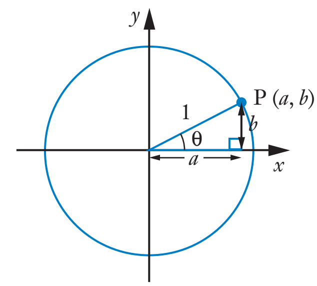
The Pythagorean identity



Consider some general point P lying on the unit circle and with coordinates (a, b), as shown in the diagram.

Applying the theorem of Pythagoras to the triangle shown we obtain the result:

$$b^2 + a^2 = 1 \qquad [1]$$

From our unit circle definition of sine and cosine it follows that $a = \cos\theta$ and $b = \sin\theta$.

Substituting these facts into [1] we obtain the Pythagorean identity:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

We call this an **identity** because the left hand side, $\sin^2\theta + \cos^2\theta$, equals the right hand side, 1, for **all** values of θ.

For example if $\theta = 10°$, $\sin^2 10° + \cos^2 10° = 1$ (by calculator),

if $\theta = 30°$,
$$\sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1,$$

if $\theta = 45°$,
$$\sin^2 45° + \cos^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1,$$

if $\theta = 125°$, $\sin^2 125° + \cos^2 125° = 1$ (by calculator), etc.

This should be compared with an equation which is true only for certain values of θ. For example, the equation $2\sin\theta = 1$ is true for certain values of θ, e.g. 30°, 150°, 390° etc., but is not true for all values of θ, e.g. $2\sin 10° \neq 1$.

In some texts the symbol $\equiv$ is used for an identity.

For example $\sin^2\theta + \cos^2\theta \equiv 1$, an identity,

but $2\sin\theta = 1$, an equation.

The Pythagorean identity can be used to help us solve some trigonometric equations, as the next example demonstrates.

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EXAMPLE 18

Solve $2\cos^2\theta + \sin\theta = 2$ for $0 \le \theta \le 360°$.

Solution

$$2\cos^2\theta + \sin\theta = 2$$

From $\sin^2\theta + \cos^2\theta = 1$ it follows that $\cos^2\theta = 1 - \sin^2\theta$.

Substituting this expression for $\cos^2\theta$ into the equation we obtain a quadratic in $\sin\theta$:

$$2(1 - \sin^2\theta) + \sin\theta = 2,$$

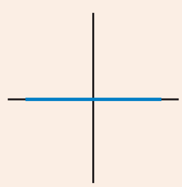
i.e. $$2 - 2\sin^2\theta + \sin\theta = 2$$

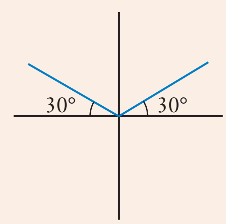
$$\sin\theta - 2\sin^2\theta = 0$$

$$\sin\theta(1 - 2\sin\theta) = 0$$

$\therefore$ either $\sin\theta = 0$ or $1 - 2\sin\theta = 0$,

$\sin\theta = 0.5$





$\theta = 0°, 180°, 360°$. $\theta = 30°, 150°$.

Thus for $0 \le \theta \le 360°$ the solutions are 0°, 30°, 150°, 180° and 360°.

As with the solving of trigonometric equations earlier in this chapter, the answers to the previous example can be obtained using the ability of some calculators to solve equations.

Again this facility is very useful but also make sure you can demonstrate your ability to solve 'trig equations' without a calculator when required to do so.

solve$(2\cdot(\cos(x))^2 + \sin(x) = 2, x) \mid 0 \le x \le 360$

x = 0 or x = 30 or x = 150 or x = 180 or x = 360

Exercise 8E

Solve the following equations for the given interval but first note the following:

- Whilst you should solve each equation **without** the assistance of a calculator you may find the information in the display below of use for some of them.
- Give exact answers where possible but when rounding is needed give answers correct to one decimal place.
- Not all of the equations require the Pythagorean identity to be used. You must decide whether its use is appropriate.

1 $\sin x = \frac{1}{4}$ for $-180° \le x \le 180°$

2 $\sin^2 x = \frac{1}{4}$ for $-\pi \le x \le \pi$

3 $\sin x = \sin^2 x + \cos^2 x$ for $0 \le x \le 2\pi$

4 $(2 \sin x - 1) \cos x = 0$ for $0 \le x \le 2\pi$

5 $\sin x + 2 \sin^2 x = 0$ for $0 \le x \le 360°$

6 $(2 \cos x + 1)(5 \sin x - 1) = 0$ for $0 \le x \le 360°$

7 $8 \sin^2 x + 4 \cos^2 x = 7$ for $0 \le x \le 2\pi$

8 $\tan^2 x + \tan x = 2$ for $-180° \le x \le 180°$

9 $5 - 4 \cos x = 4 \sin^2 x$ for $-90° \le x \le 90°$

10 $3 = 2 \cos^2 x + 3 \sin x$ for $0 \le x \le 4\pi$.

```
solve(sin(x) = 0.25, x) | 0 ≤ x ≤ 90°
                         {x = 14.47751219}
solve(sin(x) = 0.2, x) | 0 ≤ x ≤ 90°
                         {x = 11.53695903}
solve(tan(x) = 2, x) | 0 ≤ x ≤ 90°
                         {x = 63.43494882}
factor(2·y^2–3·y + 1)
                         (y – 1)·(2·y – 1)
factor(4·y^2–4·y + 1)
                         (2·y – 1)^2
```

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Angle sum and angle difference

Is the statement $\cos(A - B) = \cos A - \cos B$ true for *all* values of A and B? I.e. is the statement an identity? We can demonstrate that **it is not an identity** by considering some values for A and B.

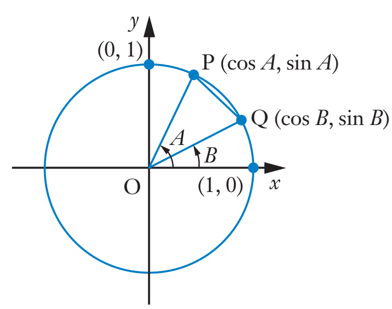
For example: If $A = 90°$ and $B = 30°$ then
$$\begin{aligned}\cos(A - B) &= \cos(90° - 30°)\\ &= \cos 60°\\ &= 0.5.\end{aligned}$$

But
$$\begin{aligned}\cos A - \cos B &= \cos 90° - \cos 30°\\ &= 0 - \frac{\sqrt{3}}{2}\\ &\neq 0.5.\end{aligned}$$

Thus $\cos(A - B) \neq \cos A - \cos B$ for these values of A and B.

Having established that $\cos(A - B)$ is not the same as $\cos A - \cos B$ can we find an expression that $\cos(A - B)$ is the same as?

Consider the points P and Q lying on the unit circle as shown in the diagram on the right. From our unit circle definition of sine and cosine the coordinates of P and Q will be as shown.



In an earlier chapter we saw that the length of the line joining two points could be found by determining

$$\sqrt{(\text{change in the } x\text{-coordinates})^2 + (\text{change in the } y\text{-coordinates})^2}$$

$$\begin{aligned}\text{Thus PQ} &= \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}\\ &= \sqrt{\cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B}\\ &= \sqrt{\cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A\cos B - 2\sin A\sin B}\\ &= \sqrt{1 + 1 - 2(\cos A\cos B + \sin A\sin B)}\\ &= \sqrt{2 - 2(\cos A\cos B + \sin A\sin B)} \qquad [\text{I}]\end{aligned}$$

However, if instead we apply the cosine rule to triangle OPQ:

$$\begin{aligned}\text{PQ} &= \sqrt{1^2 + 1^2 - 2(1)(1)\cos(A - B)}\\ &= \sqrt{2 - 2\cos(A - B)} \qquad [\text{II}]\end{aligned}$$

Comparing [I] and [II] we see that $\cos(A - B) = \cos A\cos B + \sin A\sin B$.

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$$\cos(A - B) = \cos A \cos B + \sin A \sin B \qquad [1]$$

Replacing B by $(-B)$, and remembering that $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, it follows that

$$\begin{aligned}\cos(A - (-B)) &= \cos A \cos(-B) + \sin A \sin(-B)\\ &= \cos A \cos B - \sin A \sin B\end{aligned}$$

i.e.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \qquad [2]$$

From [1],

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta\\ &= (0)\cos\theta + (1)\sin\theta\\ &= \sin\theta\end{aligned}$$

Replacing $\frac{\pi}{2} - \theta$ by ϕ (and hence θ by $\frac{\pi}{2} - \phi$) it follows that $\cos\phi = \sin\left(\frac{\pi}{2} - \phi\right)$

Thus

$$\cos\left(\frac{\pi}{2} - A\right) = \sin A$$

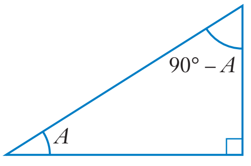
and

$$\sin\left(\frac{\pi}{2} - A\right) = \cos A$$

(These identities are sometimes referred to as the trigonometric properties of *complementarity*.)

Note

- These facts regarding A and $(90° - A)$ come as no surprise if we remember that this trigonometry for angles of any size must not contradict our initial ideas regarding the trigonometry of right triangles.



- We can now use these facts to determine expansions for $\sin(A + B)$ and for $\sin(A - B)$.

$$\begin{aligned}\sin(A - B) &= \cos[90° - (A - B)]\\ &= \cos[90° - A + B]\\ &= \cos(90° - A)\cos B - \sin(90° - A)\sin B\\ &= \sin A \cos B - \cos A \sin B\end{aligned}$$

i.e.

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \qquad [3]$$

Replacing B by $(-B)$ in [3] gives:

$$\begin{aligned}\sin(A - (-B)) &= \sin A \cos(-B) - \cos A \sin(-B)\\ &= \sin A \cos B + \cos A \sin B\end{aligned}$$

i.e.

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad [4]$$

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The identities [1], [2], [3] and [4] can be summarised as follows.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

From the previous results it follows that:

$$\begin{aligned}
\tan(A \pm B) &= \frac{\sin(A \pm B)}{\cos(A \pm B)} \\
&= \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B} \\
&= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} \pm \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} \mp \dfrac{\sin A \sin B}{\cos A \cos B}} \\
&= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

EXAMPLE 19

Determine an exact value for sin 15°.

Solution

$$\begin{aligned}
\sin 15° &= \sin(45° - 30°) \\
&= \sin 45° \cos 30° - \cos 45° \sin 30° \\
&= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2} \\
&= \frac{\sqrt{3} - 1}{2\sqrt{2}} \text{ or, with a rational denominator, } \frac{\sqrt{2}\left(\sqrt{3} - 1\right)}{4}.
\end{aligned}$$

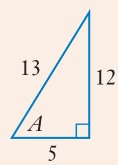
EXAMPLE 20

A and B are acute angles with $\cos A = \frac{5}{13}$ and $\sin B = \frac{24}{25}$.

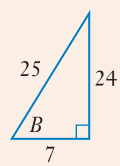
Find the exact value of $\sin(A + B)$.

Solution

If $\cos A = \frac{5}{13}$ then $\sin A = \frac{12}{13}$ (see diagram).



If $\sin B = \frac{24}{25}$ then $\cos B = \frac{7}{25}$ (see diagram).



$$\begin{aligned}\text{Thus } \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ &= \frac{12}{13}\frac{7}{25} + \frac{5}{13}\frac{24}{25} \\ &= \frac{204}{325}\end{aligned}$$

EXAMPLE 21

Solve $\cos\left(x + \frac{\pi}{4}\right) = \sqrt{2}\cos x$, for $-2\pi \le x \le 2\pi$.

Solution

$$\cos\left(x + \frac{\pi}{4}\right) = \sqrt{2}\cos x$$

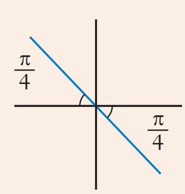
$\therefore$ $$\cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4} = \sqrt{2}\cos x$$

$$\cos x \frac{1}{\sqrt{2}} - \sin x \frac{1}{\sqrt{2}} = \sqrt{2}\cos x$$

(× by $\sqrt{2}$) $$\cos x - \sin x = 2\cos x$$

$$-\sin x = \cos x$$

(÷ by $-\cos x$) $$\tan x = -1$$



Thus for $-2\pi \le x \le 2\pi$ the solutions are $-\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$.

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Exercise 8F

Simplify each of the following.

1 $\sin 2x \cos x + \cos 2x \sin x$

2 $\cos 3x \cos x + \sin 3x \sin x$

3 $\sin 5x \cos x - \cos 5x \sin x$

4 $\cos 7x \cos x - \sin 7x \sin x$

Use the formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$ to determine exact values for each of the following.

5 $\cos 15°$

6 $\tan 15°$

7 $\sin 75°$

8 $\cos 75°$

9 $\tan 75°$

10 If $2\sin(\theta + 45°) = a\sin\theta + b\cos\theta$, find a and b.

11 If $8\cos\left(\theta - \frac{\pi}{3}\right) = c\sin\theta + d\cos\theta$, find c and d.

12 If $4\cos(\theta + 30°) = e\cos\theta + f\sin\theta$, find e and f.

13 If $\tan A = 5\sqrt{3}$ and $\tan B = -\frac{\sqrt{3}}{4}$, find (without a calculator) the value of $\tan(A + B)$.

If $\pi \le (A + B) \le 2\pi$ determine $(A + B)$.

14 A and B are acute angles with $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$. Find the following as exact values.

a $\sin(A + B)$

b $\cos(A - B)$

15 D and E are acute angles with $\sin D = \frac{7}{25}$ and $\sin E = \frac{3}{5}$. Find the following as exact values.

a $\sin(D - E)$

b $\cos(D + E)$

16 Use the expansion of $\sin(A + B)$ to prove that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

17 Use the expansion of $\sin(A \pm B)$ to prove

a $\sin(x + 2\pi) = \sin x$

b $\sin(x - 2\pi) = \sin x$

18 Use the expansion of $\cos(A + B)$ to prove that $\cos(x + 2\pi) = \cos x$.

19 Use the expansion of $\tan(A + B)$ to prove that $\tan(x + \pi) = \tan x$.

20 By writing $\tan(-x)$ as $\tan(0 - x)$, use the $\tan(A - B)$ expansion to prove that $\tan(-x) = -\tan x$.

21 A and B are both obtuse angles such that $\sin A = \frac{5}{13}$ and $\tan B = -\frac{3}{4}$. Find exact values for

a $\sin(A + B)$

b $\cos(A - B)$

c $\tan(A + B)$

Solve the following equations for the given interval.

22 $\sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6} = \frac{1}{\sqrt{2}}$ for $0 \le x \le 2\pi$.

23 $\cos x \cos 20° + \sin x \sin 20° = \frac{1}{2}$ for $0 \le x \le 360°$.

24 $\sin x \cos 70° + \cos x \sin 70° = 0.5$ for $-180° \le x \le 180°$.

25 $\sin(x + 30°) = \cos x$ for $0 \le x \le 360°$.

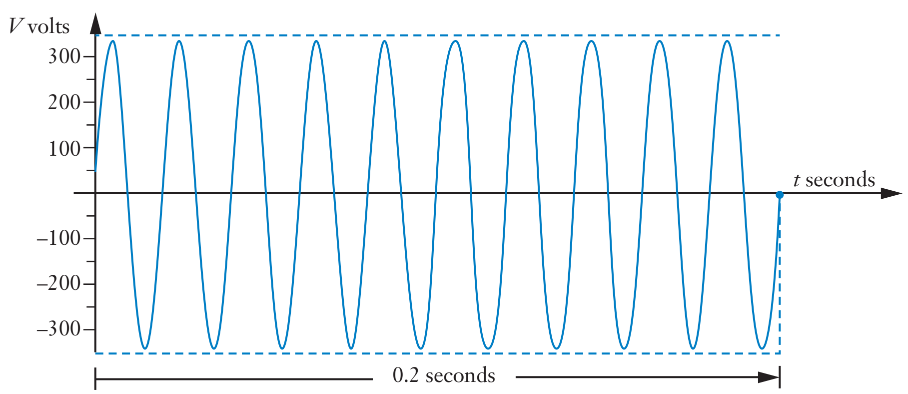
Alternating currents

An electrical current is a flow of electrical charge. In wires this electrical charge is carried by electrons. Batteries produce a steady, one directional, flow of electrons called a direct current (DC). If instead the electrons repeatedly move one way and then the other their alternating flow will still result in a flow of electrical charge, and hence a current. This is called an alternating current or AC. Many household electrical devices simply require an electrical current, they do not require the flow of electrons to always be in a certain direction. Hence for such devices an alternating current is suitable.

An alternating current is produced by an alternating voltage, which is the form of voltage that the electrical supply companies supply to most homes and businesses.

These alternating voltages are sinusoidal in nature.

Let us suppose that the voltage, V Volts, at time t seconds is as shown below:



What is the amplitude and period of this graph?

Write the equation of the graph in the form $V = a \sin bt$, with b in terms of π.

(Hint: Remember that $y = a \sin bx$ has a period of $\frac{2\pi}{b}$.)

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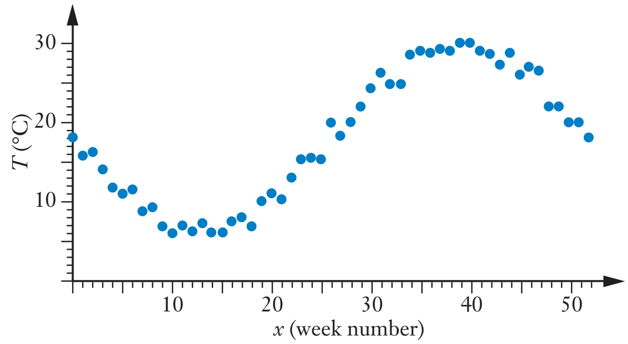
Average weekly temperatures

Records taken of the average weekly temperatures in a particular location gave rise to the following graph:



For how many of the weeks did the average weekly temperature exceed 25°C?

Suggest a suitable equation for this data. (Remember, $y = a \sin bx$ has a period of $\frac{2\pi}{b}$.)

Tidal motion

The tide height was measured at a high tide and each hour thereafter for 25 hours. The data collected gave rise to the following table:

Hours from hightide (t)	0	1	2	3	4	5	6	7	8
Height (h metres)	12.60	12.03	10.38	8.17	5.91	4.12	3.25	3.54	4.92

Hours from hightide (t)	9	10	11	12	13	14	15	16	17
Height (h metres)	7.02	9.36	11.31	12.46	12.46	11.31	9.38	7.01	4.92

Hours from hightide (t)	18	19	20	21	22	23	24	25
Height (h metres)	3.51	3.24	4.08	5.90	8.21	10.42	12.03	12.60

Plot a graph of these figures and draw a smooth curve that seems to best fit the facts.

Suggest a rule for your 'best fit' line.

Use your rule to determine the values of t, for $0 \le t \le 25$, between which the tide height is at least 5 metres.

Investigate whether or not your calculator can determine a line of best fit for the given data (look for a *regression* facility). If you are able to obtain this line of best fit from your calculator compare it to the rule you determined.

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Miscellaneous exercise eight

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

State the amplitude and period of each of the following sinusoidal functions. (Assume radian measure used.)

1 $y = 5 \sin x$

2 $y = 7 \sin x$

3 $y = -3 \sin x$

4 $y = \sin 2x$

5 $y = \sin 3x$

6 $y = \sin 0.5x$

7 $y = -3 \sin 4x$

8 $y = 4 \sin 5x$

9 $y = 2 \sin \pi x$

10 Copy and complete the following table (without the assistance of a calculator), placing appropriate **exact** values in each empty cell and all denominators rational.

θ	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{4\pi}{3}$	$\frac{7\pi}{3}$	$\frac{9\pi}{4}$	11π
Sin θ								
Cos θ								
Tan θ								

11 For each of the following pairs of lines determine whether they are parallel, perpendicular or neither of these.

a $\begin{cases} y = 3x - a \\ y = x - b \end{cases}$

b $\begin{cases} y = 0.5x + c \\ 2y = x + d \end{cases}$

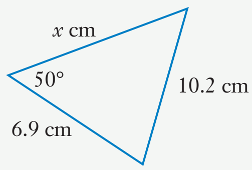
c $\begin{cases} 2y = x + e \\ y = -2x + f \end{cases}$

12 For the triangle shown on the right find x, using

a the cosine rule and the solve facility of your calculator

b the sine rule twice

giving your answer correct to one decimal place each time.



13 A triangle has sides of length 27 cm, 33 cm and 55 cm. Find the size of the smallest angle of the triangle, giving your answer to the nearest degree.

14 For each of the following, without using a calculator, write the coordinates of the points where the graph cuts or touches the x-axis. (Then check your answers with your calculator if you wish.)

a $y = (x - 2)(x - 3)(x + 2)(x + 7)$

b $y = x(x - 2)(x + 3)(x - 4)$

c $y = (x - 2)(x - 3)(x + 3)^2$

d $y = (x - 2)^4$

e $y = (x - 7)(2x^2 - 3x + 2)$

f $y = (x^2 - x - 30)(4x^2 - 8x - 21)$

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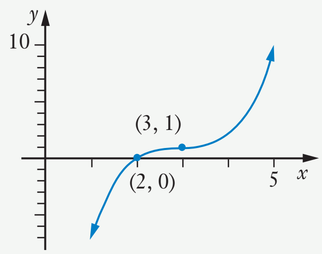
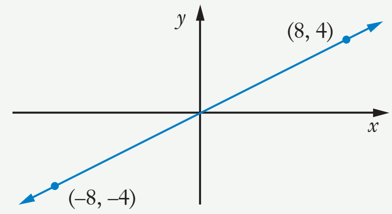
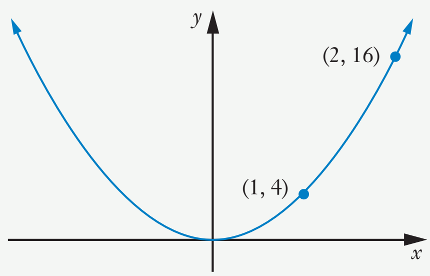
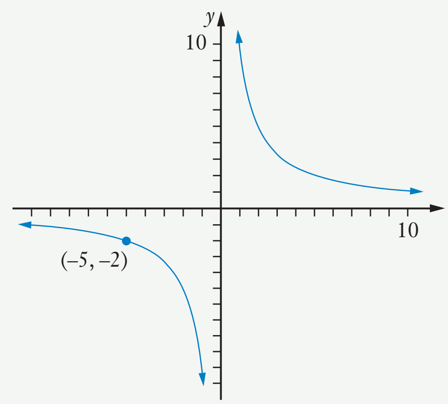
15 All of the functions f_1 to f_{12} have their graphs shown below. Find the values of k_1 to k_{26} given that they are all non-zero constants between –50 and 50.

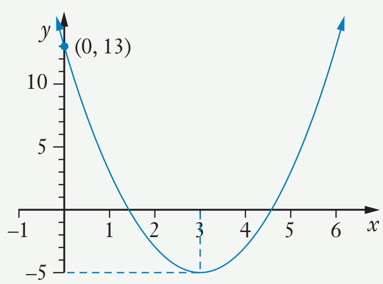
$f_1 : y = k_1x$	$f_2 : x + y + k_2 = 0$	$f_3 : y = k_3x^2$
$f_4 : y = (k_4x)^2$	$f_5 : y = k_5x^2 + k_6x + k_7$	$f_6 : xy = k_8$
$f_7 : y = k_9(x + k_{10})^3 + k_{11}$	$f_8 : y = k_{12}(x + k_{13})^2 + k_{14}$	$f_9 : y = k_{15}x^3 + k_{16}x^2 + k_{17}x + k_{18}$
$f_{10} : y = k_{19} \sin k_{20}x$	$f_{11} : y = k_{21} \cos k_{22}(x - k_{23})$	$f_{12} : y = k_{24} + k_{25} \sin k_{26}x$

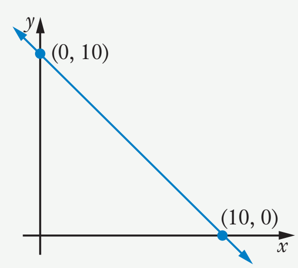


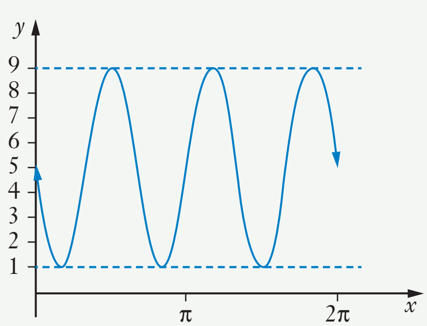


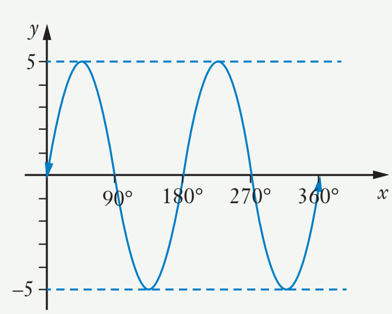












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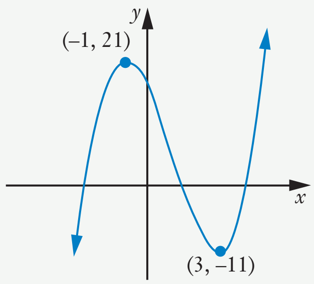
16 Given that $2x^3 + x^2 - 22x + 24 = (x - 2)(ax^2 + bx + c)$:

a determine the value of a and the value of c by inspection.

b with your answers from part **a** in place, expand $(x - 2)(ax^2 + bx + c)$ and hence determine b.

c find the coordinates of the point(s) where the graph of $y = 2x^3 + x^2 - 22x + 24$ cuts the x-axis.

17 The function $y = f(x)$, shown graphed on the right, has a maximum turning point at $(-1, 21)$ and a minimum turning point at $(3, -11)$.



State the coordinates of the maximum and minimum turning points of each of the following functions.

a $y = f(x) + 5$

b $y = f(x) - 5$

c $y = f(-x)$

d $y = -f(x)$

e $y = 3f(x)$

f $y = f(2x)$

18 **Work through this question without the assistance of a graphic calculator.**

For the graph of $\quad y = (x + 1)(x - 2)(x - 5)$

a find the coordinates of any point(s) where the curve cuts the x-axis.

b find the coordinates of any point(s) where the curve cuts the y-axis.

c if the point A(1, a) lies on the curve determine the value of a.

d if the point B(3, b) lies on the curve determine the value of b.

e if the point C(4, c) lies on the curve determine the value of c.

For the graph of $\quad y = (x - 3)^2 - 4$

f find the coordinates of the turning point and state whether it is a maximum point or a minimum point.

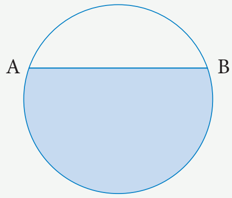
g find the coordinates of any point(s) where the curve cuts the y-axis.

h if the point D(5, d) lies on the curve determine the value of d.

i use the information from parts **a** to **h** above to produce a single sketch showing the two functions and hence estimate solutions to the equation:

$$(x + 1)(x - 2)(x - 5) = (x - 3)^2 - 4.$$

19 Determine the area of the shaded region shown on the right given that the circle has a radius of 10 cm and AB is of length 16 cm. Give your answer correct to the nearest 0.1 cm^2.



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